Geometric group theory
Example sheet 1

Lent 2023

Questions marked * are more involved.

1. Prove that F_2, the free group of rank 2, contains a subgroup isomorphic to F_∞, the free group of countably infinite rank. [*Hint: Realise F_2 as the fundamental group of a suitable space, and exhibit an appropriate covering space.*]

2. Equip $\mathbb{Z}$ with the standard metric $d(x,y) = |x-y|$. The isometry group of $\mathbb{Z}$ is called the *infinite dihedral group* and denoted by D_∞. Prove that D_∞ is isomorphic to each of the following:

   (a) $\langle s, t \mid s^2, stst \rangle$;

   (b) the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$;

   (c) the unique non-trivial semidirect product $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$.

3. Prove that the fundamental group of the Klein bottle is isomorphic to the unique non-trivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$.

4. Let Σ be the closed, orientable surface of genus 2. Exhibit a simple closed curve α on Σ so that cutting along α decomposes Σ as an amalgamated free product. Exhibit a simple closed curve β on Σ so that cutting along β decomposes Σ as an HNN extension.

5. Consider the (2,3)-Baumslag–Solitar group $BS(2,3) = \langle a, b \mid ba^2b^{-1}a^{-3} \rangle$.

   (a) Express $BS(2,3)$ as an HNN extension.

   (b) Prove that there is a surjective homomorphism ϕ from $BS(2,3)$ to itself that sends a to a^2 and b to b.

(c) Show that $[bab^{-1}, a^2]$ is in the kernel of ϕ.

6. The *Heisenberg group* is the group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}$$

of integer matrices. Prove that there is a short exact sequence

$$1 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 1\,,$$

and that this short exact sequence does not split.

7. Let G be a finitely generated group. Show that G is infinite if and only if any Cayley graph $\mathrm{Cay}(G, S)$ contains an infinite path.

8. Let F_n be the free group of rank n. Prove that $\mathrm{Aut}(F_n)$ contains a subgroup isomorphic to $\mathbb{Z}^{2n-2}$.

9. Prove that the standard Cayley tree of the free group of rank 2 admits uncountably many graph automorphisms.

10. * Find an algorithm that determines whether or not a given pair of elements g, h in the free group F_n are conjugate.

11. * Let $G = A * B$ be a free product. Let X, Y be simplicial complexes with fundamental groups A and B respectively. Construct a simplicial complex Z with fundamental group G by gluing the two ends of a closed interval I to base points in A and B. Consider an element

$$g = a_1 b_1 \ldots a_n b_n$$

of G, and suppose that $g = 1$.

(a) Consider a loop γ in Z representing g. Show that, after possibly modifying by a homotopy, γ extends to a simplicial map from a disc to Z. [*You may use the simplicial approximation theorem without proof.*]

(b) By considering the preimage of the midpoint of I in the disc, prove that some term a_i or b_i of g is trivial.

(c) Formulate a notion of reduced words for free products, and prove that every non-trivial element is equal to a unique reduced word.